On the motion of a liquid in a spheroidal cavity of a precessing rigid body

By K. STEWARTSON
Mathematics Research Center, University of Wisconsin, Madison, Wisconsin, U.S.A.†

AND P. H. ROBERTS
Yerkes Observatory, Williams Bay, Wisconsin, U.S.A.

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The flow set up in an oblate cavity of a precessing rigid body is examined under the assumptions that the ellipticity of the spheroidal boundary of the fluid is large compared with Ω/ω and that the boundary-layer thickness is small compared with the deviations of the boundary from sphericity ($\boldsymbol{\omega}$ is the angular velocity of the rigid body about the axis of symmetry, $\boldsymbol{\Omega}$ is the angular velocity with which this axis precesses).

The motion of the fluid is found by considering an initial-value problem in which the axis of rotation of the spheroid is impulsively moved at a time $t = 0$; before that time this axis is supposed to be fixed in space, the fluid and envelope turning about it as a solid body. The solution is divided into a steady motion and transients, and, by evaluating the effects of the viscous boundary layer, the transients are shown to decay with time. The steady motion which remains consists of a primary rigid-body rotation with the envelope, superimposed on which is a circulation with constant vorticity in planes perpendicular to $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\Omega})$, the streamlines being similar and similarly situated ellipses.

The possible effects of the luni-solar precession on the fluid motions in the Earth's core are discussed.

1. Introduction

Some years ago, Bondi & Lyttleton (1953) examined the motion of a liquid in a spherical cavity of a rigid body which was rotating with an angular velocity $\boldsymbol{\omega}$ about an axis through the centre of the sphere when this axis, in turn, was precessing with angular velocity $\boldsymbol{\Omega}$ ($\Omega \ll \omega$) about an axis fixed in space. The aim of their investigation was to throw some light on the effect of the precession of the Earth on the motion of the liquid core. The analytical approach used by Bondi & Lyttleton was to suppose that the fluid is Newtonian and practically inviscid. Then, in the first approximation, viscosity was neglected, and it was hoped to correct for viscosity by a thin boundary layer near the surface of the cavity. Unfortunately, on attempting to solve the steady inviscid equations, they arrived at a definite contradiction which strongly suggested to them that no steady-state motion of a permanent character is possible for the fluid.

† Present address: Department of Mathematics, The University of Durham, Durham.

The spherical cavity presents a special difficulty in the theory of rotating fluids because without viscosity no motion can be communicated to the fluid by a rotation of the boundary. The angular velocity acquired by the fluid, even if the sphere is not precessing, is due entirely to the generation of vorticity in the boundary layer at the surface of the cavity and one might expect it to take a time $O(a^2/\nu)$, where a is the radius of the cavity and ν the kinematic viscosity, for the motion of the fluid and sphere to rotate together as a rigid body. Let us suppose this time has passed, that the precessing motion is set up at $t = 0$, and that subsequently the fluid may be regarded as inviscid. Then while the solid body precesses the fluid rotates about a fixed axis, being unaffected by the motion of its envelope. The motion of the fluid is actually steady but relative to the envelope it appears to be unsteady, its axis of rotation precessing with angular velocity $-\mathbf{\Omega}$ and, when Ωt is small, this means that the velocity perturbations are proportional to Ωt. Thus the contradiction obtained by Bondi & Lyttleton does not necessarily mean that the motion of the fluid is not ultimately steady. For a real fluid, one might expect that the motion described above would occur if $\Omega a^2/\nu \gg 1$, and if, on the other hand, $\Omega a^2/\nu \ll 1$, the motion of the fluid and envelope would be virtually a rigid body precession.

This discussion strongly supports their view that the determination of the motion is likely to prove a problem of great difficulty, and consequently we thought it desirable to begin our attack by studying the related problem of the oblate spheroidal envelope which includes the sphere as a special case but proves easier because the motion of the boundary of the cavity must communicate itself in part to the fluid. This problem is not without interest in the geophysical context with which Bondi & Lyttleton were concerned because, although one can say that the core of the Earth is nearly a sphere in that

$$(a^2-b^2)/a^2 \ll 1,$$

where a and b are the semi-major and minor axes of the core, one cannot be sure of the values of

$$R_1 = (a^2-b^2)\,\omega/\Omega a^2, \quad R_2 = \Omega a^2/\nu,$$

which, with $R_3 = \omega a^2/\nu$ ($\gg 1$), control the motion of the fluid in the core relative to the rigid-body rotation. If $R_1 = 0$, the motion is critically dependent on R_2 while if R_1 is large the motion is independent of R_2 except, possibly, if $R_1 R_2 \lesssim 1$. Our investigation here is based on the assumptions that $R_1 \gg 1$, $R_1 R_2 \gg 1$; it is hoped to explore other limiting situations in a later paper. These assumptions are not unreasonable in the geophysical context since, for the surface of the Earth, $(a^2-b^2)/2a^2 \simeq 1/297$, and the value of this parameter for the core is not likely to be many orders of magnitude less. Further, $\omega/\Omega \simeq 10^7$, and ν is almost certainly much less than $10^8\,\text{cm}^2/\text{sec}$. We shall consider these geophysical questions further in § 7.

Even for an oblate spheroidal cavity containing almost inviscid fluid the determination of the flow has some unusual features. Thus the governing equations for an inviscid fluid are hyperbolic while the boundary condition is a relation between the function and its normal derivative. Such a boundary condition is appropriate to an elliptic differential equation and there are no corresponding existence and uniqueness theorems for hyperbolic equations. In fact one can construct examples of non-existence and non-uniqueness as follows.

Suppose that the governing equation is

$$\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2} = 0, \tag{1.1}$$

and that (*a*) $\phi = 0$ on the circle $x^2+y^2 = 1$, or (*b*) $\phi = 0$ on three sides of the square $|x| \leqslant 1$, $|y| \leqslant 1$ and $\phi = 1$ on the fourth side. One solution of (*a*) is

$$\phi = A(x^2+y^2-1)$$

inside the circle, where A is arbitrary. This solution can be added to any other, so that there is no unique solution of (1.1) in this case. Again, every solution of (1.1) which vanishes on three sides of a square must vanish on the fourth side so that the solution of problem (*b*) cannot exist.

Fortunately in the present problem there is a simple physical explanation of these phenomena. The non-uniqueness is associated with the existence of free oscillatory motions of the fluid in which the boundary remains fixed and the non-existence is associated with the resonance which can occur when the boundary is made to oscillate with the same period as that of one of the free oscillations of the fluid.

The difficulty about the inappropriate type of boundary condition is removed by considering motions which start from a position of relative rest. We suppose that the fluid and envelope are initially rotating as if solid with angular velocity $\boldsymbol{\omega}$ and that at time $t = 0$ the envelope is set precessing with angular velocity $\boldsymbol{\Omega}$. The equations governing the induced motion of the fluid are unsteady but the time dependence is removed on applying a Laplace transformation with parameter s and the equation is reduced to a single one of Laplace's form apart from a scaling factor which is a function of s. A general solution can be written down and, on inversion, the history of the motion can be traced. From an examination of the solution at large times we can relate the non-uniqueness to free oscillations set up by the initial motion and the non-existence to resonance.

For a spheroid the actual inviscid solution is straightforward being effectively given by Poincaré (1910); the difficulty found by Bondi & Lyttleton for $a = b$ is seen to be a resonance and explicable on the lines stated in physical terms at the beginning of this section.

From a geophysical standpoint an argument based on an initial motion is incomplete without considering the effect of dissipation which on the geological time scale might be expected to have a serious effect on the solution. In the inviscid solution predicted for a spheroid the velocity components are linear functions of position so that apparently there is no dissipation in the interior, it being confined to the boundary layer which must arise through violation of the no-slip condition. This is investigated in § 4. The inviscid motion outside consists of two parts: (i) dependent on the azimuth angle θ and due to the steady motion of the boundary, (ii) independent of θ and due to the angular velocity which the boundary acquired initially but failed to communicate immediately to the fluid. The boundary layer associated with (i) was studied by Bondi & Lyttleton but we repeat their investigation here since we are including time as a parameter. It develops a singularity on certain circles of latitude as s approaches any value on the imaginary axis between $-3i$ and i. The boundary layer associated with (ii) has also been studied earlier by Proudman (1956) and Stewartson (1957), and it develops singularities as s approaches any value on the imaginary axis between $\pm 2i$.

Bondi & Lyttleton (1953) noticed that the normal velocity on the critical circle become infinite in the limit and suggested that the flow near it was unstable leading to turbulence here and elsewhere. Although one cannot be sure without carrying out an actual experiment, more recent investigations do not support this view. The singularity on a critical line is a singularity in the boundary-layer sense, i.e. if v_n is the normal velocity just outside the boundary layer, the equations imply that $v_n = O(\nu^{\frac{1}{2}})$ almost everywhere and on approaching the critical line $\nu^{-\frac{1}{2}}v_n \to \infty$. This does not necessarily mean that $v_n \to \infty$ and indeed, in a related problem (Roberts & Stewartson 1963), a more detailed investigation of the flow properties near the critical line implied $v_n = O(\nu^{\frac{2}{5}})$ which is still small in the limit $\nu \to 0$. Further, a detailed solution of a problem related to (ii) by Stewartson (1957) elucidates the role of the critical circles as the origin of shear layers which penetrate into the fluid, transporting fluid from one part of the boundary to another via the interior and also adjusting the angular velocity of the main body of fluid when required. Free shear layers are certainly notoriously prone to instability; however, after Proudman (1956) had postulated their existence in a discussion of the flow between two concentric spheres, experiments carried out by Fultz & Moore (1962) to test his conclusions did not reveal any serious instability. In fact, in the case when the outer sphere was rotated more rapidly than the inner sphere, a shear layer of the predicted structure was observed. It was noted however that, when the inner sphere was rotated the faster, the shear layer was much broader and more diffusive due, it appeared, to the formation of a street of line vortices. But, even in this case, the main characteristics of the flow far from the shear layer were not greatly affected.

In §§ 5 and 6 the (tertiary) modification to the secondary flow induced by these boundary layers is discussed with particular reference to the oscillations produced by the initial motion of the envelope. It is shown that, of the motions dependent on θ, only the steady motion survives as $t \to \infty$. The effect of the boundary layer independent of θ is to lead to a breakdown in the tertiary flow as $t \to \infty$, and it is argued that this must mean that, relative to the boundary, the motion independent of θ must die out as $t \to \infty$ due to the communication of vorticity to the fluid via the boundary layer.

Since the steady solution dependent on θ is solely determined by the residue of the pole of the Laplace transform of the velocity at $s = 0$, it is independent of the initial condition assumed. Consequently the motion which finally develops as a result of the joint action of the moving boundary and dissipation is the same whatever the initial condition assumed, and the non-uniqueness found when the steady problem is considered is seen to be illusory, i.e. any other solutions of the inviscid problem would undergo slow changes through the action of dissipation, ultimately taking on the form described above.

2. Equations of unsteady motion

We consider a mass of incompressible fluid which occupies the whole of a rigid envelope whose internal boundary, S_0, is a surface of revolution with axis of symmetry L_B. Initially the fluid and the envelope are rotating about the axis with angular velocity $\boldsymbol{\omega}$ which is an absolute constant. At time $t = 0$, the axis L_B is set rotating with a small uniform angular velocity $\boldsymbol{\Omega}$ about an axis L_S fixed in space which intersects L_B at a point O, and which is inclined at an angle α to it. For definiteness, we shall suppose that the perturbed motion is started impulsively so that $\boldsymbol{\Omega}$ is also an absolute constant. We wish to find the subsequent motion of the fluid.

Consider a reference frame, $\mathscr{F}$, rotating with angular velocity $\boldsymbol{\Omega}$, relative to which L_B and L_S are stationary, and let the velocity of the fluid relative to $\mathscr{F}$ be

$$\mathbf{w} = \mathbf{u} + \boldsymbol{\omega} \times \mathbf{r}. \tag{2.1}$$

At $t = 0-$, $\mathbf{u} = 0$, and at $t = 0+$, $\mathbf{u} \neq 0$ in virtue of the impulsive motion of S_0, but it will be determined by the initial motion of S_0. The actual velocity, $\mathbf{v}$, of the fluid, relative to axes fixed in space and instantaneously coinciding with $\mathscr{F}$, is

$$\mathbf{v} = \mathbf{u} + \boldsymbol{\omega} \times \mathbf{r} + \boldsymbol{\Omega} \times \mathbf{r}. \tag{2.2}$$

We know that, at $t = 0+$, $\mathbf{u} + \boldsymbol{\Omega} \times \mathbf{r}$ is irrotational and uniquely determined by the motion of S_0, and hence is also known at $t = 0+$. The equations of motion relative to $\mathscr{F}$ are

$$\partial \mathbf{v}/\partial t + \boldsymbol{\Omega} \times \mathbf{v} + (\mathbf{w}\,.\,\mathrm{grad})\,\mathbf{v} = -\mathrm{grad}\,(W + p/\rho) + \nu \nabla^2 \mathbf{v}, \tag{2.3}$$

$$\mathrm{div}\,\mathbf{v} = 0, \tag{2.4}$$

where p is the pressure, ρ the density, ν the kinematic viscosity, and W the potential per unit mass of the external forces, supposed conservative, which act on the fluid. We now express (2.3) in terms of $\mathbf{u}$, obtaining

$$\partial \mathbf{u}/\partial t + 2(\boldsymbol{\omega} + \boldsymbol{\Omega}) \times \mathbf{u} - (\mathbf{u} + \boldsymbol{\omega} \times \mathbf{r}) \times \mathrm{curl}\,\mathbf{u}$$
$$= -\mathrm{grad}\,[W + (p/\rho) - \tfrac{1}{2}\{(\boldsymbol{\omega} + \boldsymbol{\Omega}) \times \mathbf{r}\}^2 + \mathbf{u}\,.\,(\boldsymbol{\omega} \times \mathbf{r}) + \tfrac{1}{2}\mathbf{u}^2] + \nu \nabla^2 \mathbf{u} + \mathbf{r} \times (\boldsymbol{\Omega} \times \boldsymbol{\omega}),$$

whence, on neglecting squares and products of $\mathbf{u}$ and $\boldsymbol{\Omega}$, we find

$$\partial \mathbf{u}/\partial t + 2\boldsymbol{\omega} \times \mathbf{u} - (\boldsymbol{\omega} \times \mathbf{r}) \times \mathrm{curl}\,\mathbf{u} = -\omega\,\mathrm{grad}\,V' + \nu \nabla^2 \mathbf{u} - 2(\boldsymbol{\Omega}\,.\,\mathbf{r})\,\boldsymbol{\omega}, \tag{2.5}$$

where

$$\omega V' = W + (p/\rho) + \tfrac{1}{2}\mathbf{v}^2 - (\boldsymbol{\omega} \times \mathbf{r})^2 + (\boldsymbol{\omega}\,.\,\mathbf{r})\,(\boldsymbol{\Omega}\,.\,\mathbf{r}) - 2(\boldsymbol{\Omega}\,.\,\boldsymbol{\omega})\,\mathbf{r}^2. \tag{2.6}$$

This equation differs from that given by Bondi & Lyttleton (1953), viz.

$$[\partial \mathbf{u}/\partial t] + 2\boldsymbol{\omega} \times \mathbf{u} - (\boldsymbol{\omega} \times \mathbf{r}) \times \mathrm{curl}\,\mathbf{u} = -\omega\,\mathrm{grad}\,V + \nu \nabla^2 \mathbf{u} - (\boldsymbol{\Omega}\,.\,\mathbf{r})\,\boldsymbol{\omega}, \tag{2.7}$$

where

$$\omega V = W + (p/\rho) + \tfrac{1}{2}\mathbf{v}^2 - (\boldsymbol{\omega} \times \mathbf{r})^2 + 2(\boldsymbol{\omega}\,.\,\mathbf{r})\,(\boldsymbol{\Omega}\,.\,\mathbf{r}) - \tfrac{5}{2}(\boldsymbol{\Omega}\,.\,\boldsymbol{\omega})\,\mathbf{r}^2.$$

The first term in (2.7) in brackets was omitted by Bondi & Lyttleton because they assumed that the motion is steady. Although both V and the forcing term in (2.7) are incorrect, in the present problem only the absence of the factor 2 in the forcing term is of significance. We shall show (§ 3) that this factor is crucial for obtaining a consistent solution of the problem when S_0 is a sphere. The equation of continuity (2.4) is equivalent to

$$\operatorname{div} \mathbf{u} = 0. \tag{2.8}$$

Now choose a system of cylindrical polar co-ordinates (r, θ, z) in $\mathscr{F}$ with L_B as the z-axis, r measuring the distance of a typical point $P(\mathbf{r})$ from L_B, and θ being the angle between the plane defined by P and L_B and the plane defined by L_S and L_B. Then

$$(\mathbf{\Omega} . \mathbf{r})\, \boldsymbol{\omega} = (0, 0, -\Omega\omega z\cos\alpha - \Omega\omega r\sin\alpha\, \cos\theta), \tag{2.9}$$

where $\Omega = -|\mathbf{\Omega}|$ is the (retrograde) angular velocity of precession.

To begin with, neglect viscosity, and denote the velocity by $\mathbf{u}_0$ with components $(u_{0r}, u_{0\theta}, u_{0z})$ respectively along the directions of r, θ and z increasing; (2.5) and (2.8) are then equivalent to

$$\left.\begin{aligned}
\frac{\partial u_{0r}}{\partial t} + \omega\frac{\partial u_{0r}}{\partial \theta} - 2\omega u_{0\theta} &= -\omega\frac{\partial V''}{\partial r},\\
\frac{\partial u_{0\theta}}{\partial t} + \omega\frac{\partial u_{0\theta}}{\partial \theta} + 2\omega u_{0r} &= -\frac{\omega}{r}\frac{\partial V''}{\partial \theta},\\
\frac{\partial u_{0z}}{\partial t} + \omega\frac{\partial u_{0z}}{\partial \theta} &= -\omega\frac{\partial V''}{\partial z} + 2\omega\Omega r\sin\alpha\, \cos\theta,\\
\frac{1}{r}\frac{\partial}{\partial r}(r u_{0r}) + \frac{1}{r}\frac{\partial u_{0\theta}}{\partial \theta} + \frac{\partial u_{0z}}{\partial z} &= 0,
\end{aligned}\right\} \tag{2.10}$$

where $$V'' = V' - \Omega z^2 \cos\alpha - r u_{0\theta}.$$

The boundary conditions require that the normal component of the fluid velocity is zero on S_0, and also that, at $t = 0+$, $\mathbf{u} + \mathbf{\Omega}\times\mathbf{r}$ is irrotational with velocity potential ϕ_i (say).

In order to solve (2.10) it is convenient to take the Laplace transform of the dependent variables with respect to time denoting the result by an asterisk, e.g.

$$u^*_{0r}(r, \theta, z, s) = \int_0^\infty e^{-\omega st} u_{0r}(r, \theta, z, t)\, dt. \tag{2.11}$$

The governing equations then reduce to

$$\left.\begin{aligned}
\left(\frac{\partial}{\partial \theta} + s\right) u^*_{0r} - 2u^*_{0\theta} &= -\frac{\partial V''^*}{\partial r} + \frac{1}{\omega}(u_{0r})_i,\\
\left(\frac{\partial}{\partial \theta} + s\right) u^*_{0\theta} + 2u^*_{0r} &= -\frac{1}{r}\frac{\partial V''^*}{\partial \theta} + \frac{1}{\omega}(u_{0\theta})_i,\\
\left(\frac{\partial}{\partial \theta} + s\right) u^*_{0z} &= -\frac{\partial V''^*}{\partial z} + \frac{1}{\omega}(u_{0z})_i + \frac{2\Omega\sin\alpha}{\omega s} r\cos\theta,
\end{aligned}\right\} \tag{2.12}$$

where the suffix i denotes initial conditions. Since, at $t = 0+$, $\mathbf{u} + \mathbf{\Omega}\times\mathbf{r}$ is derivable from the potential ϕ_i, equations (2.12) may be simplified, on writing

$$V^* = V''^* - \omega^{-1}\phi_i + \Omega\omega^{-1} rz\sin\alpha\, \sin\theta, \tag{2.13}$$

to

$$\left.\begin{aligned}\left(\frac{\partial}{\partial\theta}+s\right)u_{0r}^*-2u_{0\theta}^* &= -\frac{\partial V^*}{\partial r},\\ \left(\frac{\partial}{\partial\theta}+s\right)u_{0\theta}^*+2u_{0r}^* &= -\frac{1}{r}\frac{\partial V^*}{\partial\theta}+\frac{\Omega}{\omega}r\cos\alpha,\\ \left(\frac{\partial}{\partial\theta}+s\right)u_{0z}^* &= -\frac{\partial V^*}{\partial z}+\frac{2\Omega}{\omega s}r\sin\alpha(\cos\theta+s\sin\theta),\end{aligned}\right\} \tag{2.14}$$

and the equation of continuity to

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_{0r}^*)+\frac{1}{r}\frac{\partial u_{0\theta}^*}{\partial\theta}+\frac{\partial u_{0z}^*}{\partial z} = 0. \tag{2.15}$$

A further simplification is obtained by writing

$$\left.\begin{aligned}u_{0r}^* &= \mathscr{R}(\overline{u}_{0r}e^{i\theta}), & u_{0\theta}^* &= (\Omega r\cos\alpha/\omega s)+\mathscr{R}(\overline{u}_{0\theta}e^{i\theta}),\\ u_{0z}^* &= \mathscr{R}(\overline{u}_{0z}e^{i\theta}), & V^* &= (\Omega r^2\cos\alpha/\omega s)+\mathscr{R}(\overline{V}e^{i\theta}),\end{aligned}\right\} \tag{2.16}$$

where $\overline{u}_{0r}$, $\overline{u}_{0\theta}$, $\overline{u}_{0z}$, and $\overline{V}$ are independent of θ. On substituting into (2.14) and (2.15), we obtain

$$\left.\begin{aligned}\overline{u}_{0r} &= -\frac{1}{(s+i)^2+4}\left[(s+i)\frac{\partial\overline{V}}{\partial r}+\frac{2i\overline{V}}{r}\right],\\ \overline{u}_{0\theta} &= -\frac{i}{(s+i)^2+4}\left[2i\frac{\partial\overline{V}}{\partial r}+(s+i)\frac{\overline{V}}{r}\right],\\ \overline{u}_{0z} &= -\frac{2i\Omega\sin\alpha}{\omega s}r-\frac{1}{s+i}\frac{\partial\overline{V}}{\partial z},\end{aligned}\right\} \tag{2.17}$$

where $\overline{V}$ satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\overline{V}}{\partial r}\right)-\frac{\overline{V}}{r^2}+\frac{[(s+i)^2+4]}{(s+i)^2}\frac{\partial^2\overline{V}}{\partial z^2} = 0. \tag{2.18}$$

If S_0 is given by $f(r,z) = 1$, the boundary condition to be satisfied is

$$\overline{u}_{0r}\frac{\partial f}{\partial r}+\overline{u}_{0z}\frac{\partial f}{\partial z} = 0 \quad \text{at} \quad f = 1. \tag{2.19}$$

3. The secondary inviscid flow for a spheroid

The problem posed by equations (2.17) to (2.19) can be reduced to a solution of Laplace's equation on writing $[(s+i)^2+4]/(s+i)^2 = \xi^2$, and replacing z by ξz. We can expect, therefore, that it can always be solved for the boundary condition (2.19), which is equivalent to a linear relation between $\overline{V}$, its normal derivative, and possibly also its tangential derivative. Having obtained the solution, the ultimate flow may be found by letting $s \to 0$, and taking note of the poles and branch points in the solution in the half plane $\mathscr{R}(s) \geqslant 0$.

In this paper we are particularly concerned with the properties of the solution when the envelope S_0 is the oblate spheroid

$$\frac{r^2}{a^2}+\frac{z^2}{b^2} = 1 \quad (a \geqslant b), \tag{3.1}$$

of which Bondi & Lyttleton considered the special case $a = b$. For the boundary (3.1), (2.19) becomes

$$\frac{r}{a^2}\bar{u}_{0r}+\frac{z}{b^2}\bar{u}_{0z} = 0, \tag{3.2}$$

and the appropriate solution of (2.18) is found by writing

$$\bar{V} = Arz, \tag{3.3}$$

where A is independent of r and z. Substituting into (2.17) we have

$$\bar{u}_{0r} = -\frac{Az}{(s-i)}, \quad \bar{u}_{0\theta} = -\frac{iAz}{(s-i)}, \quad \bar{u}_{0z} = -\frac{iA}{(s+i)}-\frac{2i\Omega r\sin\alpha}{\omega s}. \tag{3.4}$$

Hence, from (3.2),

$$A = -\frac{2i\Omega a^2\sin\alpha(s^2+1)}{\omega s[a^2(s-i)+b^2(s+i)]}. \tag{3.5}$$

Inverting the solution with respect to s, and using (2.16), we have

$$u_{0r} = \frac{2\Omega za^2\sin\alpha}{a^2-b^2}\left[\sin\theta-\frac{2a^2}{a^2+b^2}\sin(\theta+\omega kt)\right], \tag{3.6}$$

$$u_{0\theta} = \frac{2\Omega za^2\sin\alpha}{a^2-b^2}\left[\cos\theta-\frac{2a^2}{a^2+b^2}\cos(\theta+\omega kt)\right]+\Omega r\cos\alpha, \tag{3.7}$$

$$u_{0z} = -\frac{2\Omega rb^2\sin\alpha}{a^2-b^2}\left[\sin\theta-\frac{2a^2}{a^2+b^2}\sin(\theta+\omega kt)\right], \tag{3.8}$$

where $$k = (a^2-b^2)/(a^2+b^2).$$

According to (3.6) to (3.8), the secondary motion of the fluid consists of two parts: one is essentially due to the initial motion of the boundaries; the other is an ultimately steady motion. The first part can itself be divided into two components. One is a rigid body rotation of angular velocity $\Omega\cos\alpha$ about L_B which the rotation of the boundary *failed* to communicate to the fluid. The other is a free oscillation of the fluid. The steady part of the motion consists of the rigid body rotation $\boldsymbol{\omega}$ (cf. equation (2.1)), and a circulation (given by the terms in $\sin\theta$, $\cos\theta$ and $\sin\theta$ in (3.6), (3.7) and (3.8), respectively) in planes perpendicular to L_S the streamlines being similar and similarly situated ellipses. The vorticity of this circulation is constant and equal to $-2\Omega\sin\alpha(a^2+b^2)/(a^2-b^2)$. We shall show below that, if the fluid has a small viscosity, the effect of viscous dissipation in the boundary layer is to damp out all motions except the rigid body rotation and the circulation in planes perpendicular to L_S. Since the choice of initial conditions affects the remaining terms only, it follows that the ultimate motion of the fluid is unique and that the effect of a small viscosity is to convert an improperly posed mathematical problem into a properly posed one.

In the case of a sphere $a = b$, the relevant periods of free oscillation of the fluid are infinite, so that a resonance develops and

$$\left.\begin{aligned} u_{0r} &= -\Omega z\sin\alpha(\sin\theta+\omega t\cos\theta), \\ u_{0\theta} &= \Omega z\sin\alpha(\cos\theta+\omega t\sin\theta)+\Omega r\cos\alpha, \\ u_{0z} &= \Omega r\sin\alpha(\sin\theta+\omega t\cos\theta), \end{aligned}\right\} \tag{3.9}$$

or, in vector notation,

$$\mathbf{u} = -\mathbf{\Omega}\times\mathbf{r} - t(\mathbf{\Omega}\times\mathbf{\omega})\times\mathbf{r}. \tag{3.10}$$

The physical interpretation of this result is that, as one expects, the motion of a spherical boundary exerts no influence on the motion of the fluid and hence, relative to the frame $\mathscr{F}$, the axis of rotation of the fluid rotates with angular velocity $-\mathbf{\Omega}$ about L_S. Consequently, after time t ($\Omega t \ll 1$), the fluid rotates about an axis making an angle

$$|\mathbf{\Omega}t\times\mathbf{\omega}|/|\mathbf{\omega}|,$$

with L_B, and this is indicated by (3.10). It is noted that, on substituting (3.10) into (2.7), the equations of motion are satisfied identically, and so any change in the motion of the fluid must be initiated through a boundary layer arising from the non-satisfaction of the requirement that $\mathbf{u} = 0$ on the boundary S_0. Had the equation (2.7), required by Bondi & Lyttleton, been used, t would be replaced by $\frac{1}{2}t$ in (3.10), and the new formula for $\mathbf{u}$ would not be consistent with this physical argument.

4. The boundary layer

The inviscid solution obtained in the previous section satisfies not only the inviscid equations but also the viscous equations (2.5). Viscosity therefore manifests itself only through the fact that this solution does not satisfy the viscous boundary condition $\mathbf{u} = 0$ on the spheroidal envelope S_0, and leads to a boundary layer whose thickness we can anticipate to be $O(\nu^{\frac{1}{2}})$ if ν is small where the adjustments in the tangential components of velocity are made. We shall calculate this boundary layer in the present section and its effect on the inviscid flow in the interior will be discussed in the following sections.

Let us write

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1, \tag{4.1}$$

where $\mathbf{u}$ is the actual velocity of the fluid at any point, $\mathbf{u}_0$ is the velocity if $\nu = 0$, and $\mathbf{u}_1$ is the correction due to viscosity. Then, to order $\nu^{\frac{1}{2}}$, $\mathbf{u}_1$ is the contribution from the boundary layer to $\mathbf{u}$. Introduce spheroidal polar co-ordinates $(\lambda_1, \mu_1, \theta)$, where λ_1 and μ_1 are defined by

$$r = (\lambda_1^2 + c^2)^{\frac{1}{2}}(1-\mu_1^2)^{\frac{1}{2}}, \quad z = \lambda_1\mu_1, \tag{4.2}$$

$c^2 = a^2 - b^2$ and the spheroid S_0 is given by $\lambda_1 = b$. Then, using (3.6) to (3.8), we see that the boundary condition $\mathbf{u} = 0$ requires that, at $\lambda_1 = b$,

$$\left.\begin{aligned} u_{1\lambda_1} &= 0, \\ u_{1\mu_1} &= \frac{2\Omega ab\sin\alpha}{a^2-b^2}\left[\sin\theta - \frac{2a^2}{a^2+b^2}\sin(\theta+\omega kt)\right](b^2+\mu_1^2c^2)^{\frac{1}{2}}, \\ u_{1\theta} &= -\frac{2\Omega a^2 b\sin\alpha}{a^2-b^2}\left[\cos\theta - \frac{2a^2}{a^2+b^2}\cos(\theta+\omega kt)\right]\mu_1 - \Omega a\cos\alpha(1-\mu_1^2)^{\frac{1}{2}}. \end{aligned}\right\} \tag{4.3}$$

Beyond the boundary layer, when $(b-\lambda_1)\nu^{\frac{1}{2}}$ is large and positive, $\mathbf{u}_1 = o(1)$; we shall then be particularly interested in $u_{1\lambda}$ which is $O(\nu^{\frac{1}{2}})$ and, unlike the other components of velocity, is not exponentially small. Consequently it engenders a tertiary flow though the interior of S_0. From the boundary condition (4.3) the flow in the boundary layer may be divided into two parts one of which ($\mathbf{u}_2$) is dependent on θ, and one ($\mathbf{u}_3$) which is independent of θ. We consider the part dependent on θ first.

Write the Laplace transform of $\mathbf{u}_2$ with respect to ωt as

$$\mathscr{R}(\overline{\mathbf{u}}_2 e^{i\theta}), \tag{4.4}$$

and in the boundary layer make the conventional assumption that the operator $\partial/\partial\lambda_1 = O(\nu^{-\frac{1}{2}})$ while $\partial/\partial\mu_1$ and $\partial/\partial\theta$ are $O(1)$. Then as is usual with Ekman flows, in the governing equations θ and μ_1 may be taken constant and λ_1 may be replaced by b except when it appears in $\partial/\partial\lambda_1$. The governing equations reduce to

$$(s+i)\,\overline{u}_{2\mu_1} + \frac{2\mu_1 a}{(b^2+c^2\mu_1^2)^{\frac{1}{2}}}\,\overline{u}_{2\theta} = \frac{a^2\nu}{\omega(b^2+c^2\mu_1^2)}\frac{\partial^2\overline{u}_{2\mu_1}}{\partial\lambda_1^2}, \tag{4.5}$$

$$(s+i)\,\overline{u}_{2\theta} - \frac{2\mu_1 a}{(b^2+c^2\mu_1^2)^{\frac{1}{2}}}\,\overline{u}_{2\mu_1} = \frac{a^2\nu}{\omega(b^2+c^2\mu_1^2)}\frac{\partial^2\overline{u}_{2\theta}}{\partial\lambda_1^2}, \tag{4.6}$$

$$a(b^2+c^2\mu_1^2)^{\frac{1}{2}}\frac{\partial\overline{u}_{2\lambda_1}}{\partial\lambda_1} + \frac{\partial}{\partial\mu_1}\left[(1-\mu_1^2)^{\frac{1}{2}}(b^2+c^2\mu_1^2)^{\frac{1}{2}}\,\overline{u}_{2\mu_1}\right] + \frac{i\overline{u}_{2\theta}(b^2+c^2\mu_1^2)}{a(1-\mu_1^2)^{\frac{1}{2}}} = 0, \tag{4.7}$$

and the boundary conditions to

$$\overline{u}_{2\lambda_1} = 0, \quad \overline{u}_{2\mu_1} = \frac{2i\Omega ab\sin\alpha(b^2+c^2\mu_1^2)^{\frac{1}{2}}(s+i)}{\omega s[s(a^2+b^2)-i(a^2-b^2)]}, \quad \overline{u}_{2\theta} = \frac{2\Omega a^2 b\sin\alpha(s+i)\,\mu_1}{\omega s[s(a^2+b^2)-i(a^2-b^2)]},$$

at $\lambda_1 = b$. Also $\overline{u}_{2\mu_1}$ and $\overline{u}_{2\theta} \to 0$ as $(b-\lambda_1)\,\nu^{\frac{1}{2}} \to \infty$.

Integrating (4.5), (4.6) we have

$$\left.\begin{aligned} u_{2\mu_1} &= \frac{\Omega abi\sin\alpha(s+i)}{\omega s[s(a^2+b^2)-i(a^2-b^2)]} \\ &\quad\times\{[a\mu_1+\sqrt{(b^2+c^2\mu_1^2)}]\,e^{\delta_2(\lambda_1-b)} - [a\mu_1-\sqrt{(b^2+c^2\mu_1^2)}]\,e^{\delta_1(\lambda_1-b)}\}, \\ u_{2\theta} &= \frac{\Omega ab\sin\alpha(s+i)}{\omega s[s(a^2+b^2)-i(a^2-b^2)]} \\ &\quad\times\{[a\mu_1+\sqrt{(b^2+c^2\mu_1^2)}]\,e^{\delta_2(\lambda_1-b)} + [a\mu_1-\sqrt{(b^2+c^2\mu_1^2)}]\,e^{\delta_1(\lambda_1-b)}\}, \end{aligned}\right\} \tag{4.8}$$

where

$$\left.\begin{aligned} \delta_2^2 &= \frac{\omega}{\nu a^2}(b^2+c^2\mu_1^2)\left[s+i-\frac{2i\mu_1 a}{\sqrt{(b^2+c^2\mu_1^2)}}\right], \\ \delta_1^2 &= \frac{\omega}{\nu a^2}(b^2+c^2\mu_1^2)\left[s+i+\frac{2i\mu_1 a}{\sqrt{(b^2+c^2\mu_1^2)}}\right], \end{aligned}\right\} \tag{4.9}$$

and the signs of the square root of the expression for δ_1 and δ_2 are decided by requiring that, when s is large, real and positive, δ_1 and δ_2 are both positive. The value of $u_{2\lambda_1}$ now follows by the direct integration of (4.7) so that all properties of the boundary layer are now formally known. It is noted that the assumptions on which this boundary-layer theory is built are consistent provided only that δ_1 and δ_2 do not vanish. Exceptional cases arise only if s is purely imaginary and $|s| < 3$, when the assumptions break down on certain circles $\mu_2 = \text{const}$. Conceivably this may be serious at large times since singularities occur in the s-plane on the imaginary axis; on inverting we could get contributions to $\mathbf{u}_2$ of order

$t^n e^{is_0 t}$ $(n > 0)$. However, it is believed that this does not happen. In a closely related problem (Roberts & Stewartson 1963) the flow has been studied in the neighbourhood of these lines, and it has been shown that the main effect is to change the thickness of the layer from $O(\nu^{\frac{1}{2}})$ to $O(\nu^{\frac{2}{5}})$. Adapting the argument to the present problem similar results are obtained, and it is found that, in the neighbourhood of the singular lines, $u_{2\lambda_1} = O(\nu^{\frac{1}{5}})$, instead of $O(\nu^{\frac{1}{2}})$ as elsewhere in the boundary layer. Further, on computing the contribution from the neighbourhood of these lines to the flow in the deep interior of the spheroid, we find it to be $O(\nu^{\frac{3}{5}})$ as against $O(\nu^{\frac{1}{2}})$ from the rest of the boundary layer.

Of particular interest is the value of $\bar{u}_{2\lambda_1}$ just outside the boundary layer, i.e. when $(b-\lambda_1)\nu^{\frac{1}{2}}$ is large, but $(b-\lambda_1)$ is small. This is obtained by integrating (4.7) with respect to λ_1 from $-\infty$ to b and using (4.9). Denoting the value of $\bar{u}_{2\lambda_1}$ by $\bar{u}_{4\lambda_1}(b,\mu_1,\theta)$ we have

$$\begin{aligned} a(b^2+c^2\mu_1^2)^{\frac{1}{2}}\,\bar{u}_{4\lambda_1}(b,\mu_1,\theta) &= \frac{\Omega abi\sin\alpha(s+i)}{\omega s[s(a^2+b^2)-i(a^2-b^2)]}\left[\frac{\partial}{\partial\mu_1}\left\{(1-\mu_1^2)^{\frac{1}{2}}(b^2+c^2\mu_1^2)^{\frac{1}{2}}\right.\right. \\ &\quad\times\left.\left.\left[\frac{a\mu_1+\surd(b^2+c^2\mu_1^2)}{\delta_2}-\frac{a\mu_1-\surd(b^2+c^2\mu_1^2)}{\delta_1}\right]\right\}+\frac{(b^2+c^2\mu_1^2)}{a(1-\mu_1^2)^{\frac{1}{2}}}\right. \\ &\quad\times\left.\left\{\frac{a\mu_1+(b^2+c^2\mu_1^2)^{\frac{1}{2}}}{\delta_2}+\frac{a\mu_1-(b^2+c^2\mu_1^2)^{\frac{1}{2}}}{\delta_1}\right\}\right]. \end{aligned} \qquad (4.10)$$

Now let us consider the second part ($\mathbf{u}_3$) of the boundary layer due to that part of (4.3) independent of θ. Let the Laplace transform of $\mathbf{u}_3$ with respect to ωt be $\mathbf{u}_3^*$. Then at $\lambda_1 = b$

$$u_{3\lambda_1}^* = 0, \quad u_{3\mu_1}^* = 0, \quad u_{3\theta}^* = -\Omega a\cos\alpha(1-\mu_1^2)^{\frac{1}{2}}/\omega s, \qquad (4.11)$$

and by an analogous argument to that given above the equations governing the flow in the boundary layer reduce to

$$su_{3\mu_1}^* + \frac{2\mu_1 a}{(b^2+c^2\mu_1^2)^{\frac{1}{2}}}u_{3\theta}^* = \frac{a^2\nu}{\omega(b^2+c^2\mu_1^2)}\frac{\partial^2 u_{3\mu_1}^*}{\partial\lambda_1^2}, \qquad (4.12)$$

$$su_{3\theta}^* - \frac{2\mu_1 a}{(b^2+c^2\mu_1^2)^{\frac{1}{2}}}u_{3\mu_1}^* = \frac{a^2\nu}{\omega(b^2+c^2\mu_1^2)}\frac{\partial^2 u_{3\theta}^*}{\partial\lambda_1^2}, \qquad (4.13)$$

$$a(b^2+c^2\mu_1^2)^{\frac{1}{2}}\frac{\partial u_{3\lambda_1}^*}{\partial\lambda_1} + \frac{\partial}{\partial\mu_1}[(1-\mu_1^2)^{\frac{1}{2}}(b^2+c^2\mu_1^2)^{\frac{1}{2}}u_{3\mu_1}^*] = 0. \qquad (4.14)$$

Integrating (4.12) and (4.13) we have

$$\left.\begin{aligned} u_{3\mu_1}^* &= -\frac{\Omega ai\cos\alpha(1-\mu_1^2)^{\frac{1}{2}}}{2\omega s}[e^{\delta_3(\lambda_1-b)}-e^{\delta_4(\lambda_1-b)}], \\ u_{3\theta}^* &= -\frac{\Omega a\cos\alpha(1-\mu_1^2)^{\frac{1}{2}}}{2\omega s}[e^{\delta_3(\lambda_1-b)}+e^{\delta_4(\lambda_1-b)}], \end{aligned}\right\} \qquad (4.15)$$

where

$$\left.\begin{aligned} \delta_3^2 &= \frac{\omega}{\nu a^2}(b^2+c^2\mu_1^2)\left(s-\frac{2i\mu_1 a}{(b^2+c^2\mu_1^2)^{\frac{1}{2}}}\right), \\ \delta_4^2 &= \frac{\omega}{\nu a^2}(b^2+c^2\mu_1^2)\left(s+\frac{2i\mu_1 a}{(b^2+c^2\mu_1^2)^{\frac{1}{2}}}\right), \end{aligned}\right\} \qquad (4.16)$$

and δ_3 and δ_4 are real and positive when s is real, positive and large. The consistency of this solution is subject to the same qualifications as (4.10). Of particular interest here too is the value of $u^*_{3\lambda_1}$ just outside the boundary layer, and denoting it by $u^*_{5\lambda_1}(b, \mu_1, \theta)$ we have

$$a(b^2+c^2\mu_1^2)^{\frac{1}{2}}\, u^*_{5\lambda_1} = -\frac{\Omega a i \cos\alpha}{2\omega s}\frac{\partial}{\partial\mu_1}\left[(b^2+c^2\mu_1^2)^{\frac{1}{2}}(1-\mu_1^2)^{\frac{1}{2}}\left(\frac{1}{\delta_3}-\frac{1}{\delta_4}\right)\right]. \quad (4.17)$$

5. The tertiary inviscid flow (i)

The boundary-layer solution obtained in § 4 shows that there must be a normal velocity at the edge of the layer (i.e. as $\lambda_1 \to b-$ on the inviscid scale) of order $\nu^{\frac{1}{2}}$ and given by the sum of (4.10) and (4.17). In turn this must induce a tertiary inviscid flow throughout the interior of the spheroid, i.e. a flow governed by (2.14) and (2.15) excluding the forcing terms, and satisfying (2.19) with a right-hand side effectively equal to the sum of (4.10) and (4.17) (instead of being zero). Since (4.10) is dependent on θ while (4.17) is not, it is convenient to treat their contributions separately and in this section we shall consider the consequences of (4.10). In § 6 below we shall consider the consequences of (4.17).

Denoting the corresponding velocity in the spheroid by $\mathbf{u}_4$ we have, from (2.17), that $\mathbf{u}_4$ can be expressed in terms of a scalar $\overline{V}_4$ which satisfies (2.18), the forcing term in (2.17) being again set equal to zero. Hence, after stretching the z co-ordinate, $\overline{V}_4$ satisfies Laplace's equation and the appropriate solution can be formally written down. Let

$$r = \frac{2}{[(s+i)^2+4]^{\frac{1}{2}}}(1-\mu_4^2)^{\frac{1}{2}}(\lambda_4^2+\gamma_4^2)^{\frac{1}{2}}, \quad z = \frac{2}{(s+i)}\mu_4\lambda_4, \quad (5.1)$$

where
$$\gamma_4^2 = a^2+\tfrac{1}{4}c^2(s+i)^2. \quad (5.2)$$

In terms of μ_4 and λ_4 the spheroid is given by

$$\lambda_4 = \tfrac{1}{2}(s+i)\,b \quad (5.3)$$

and on the spheroid $\mu_4 = \mu_1 = \mu$. The most general acceptable form for $\overline{V}_4$ is

$$\overline{V}_4 = (1-\mu_4^2)^{\frac{1}{2}}(\lambda_4^2+\gamma_4^2)^{\frac{1}{2}}\sum_{n=1}^{\infty} A_n P_n'(i\lambda_4/\gamma_4)\,P_n'(\mu_4), \quad (5.4)$$

the P_n being Legendre polynomials and the A_n constants to be found. Further, the boundary condition associated with (4.10), viz.

$$\frac{r}{a^2}u_{4r}+\frac{z}{b^2}u_{4z} = \left(\frac{r^2}{a^4}+\frac{z^2}{b^4}\right)^{\frac{1}{2}} u_{4\lambda_1}(b,\mu,\theta),$$

becomes, in terms of $\overline{V}_4$,

$$\frac{a}{2}\frac{\partial\overline{V}_4}{\partial\lambda_4}+\frac{2ib}{a[(s+i)^2+4]}\overline{V}_4 = (b^2+c^2\mu^2)^{\frac{1}{2}}\,\overline{u}_{4\lambda_1} \quad (5.5)$$

at $\lambda_4 = \frac{1}{2}(s+i)\,b$. It follows from (4.10) that $\bar{u}_{4\lambda_1}$ is an odd function of μ and behaves like $(1-\mu^2)^{\frac{1}{2}}$ near $\mu = \pm 1$. Consequently n is even in (5.4), and the various A_n are determined from the equation

$$A_n\left\{\frac{a}{2}\frac{\partial}{\partial\lambda_4}\left[(\lambda_4^2+\gamma_4^2)^{\frac{1}{2}}P_n'\left(\frac{i\lambda_4}{\gamma_4}\right)\right]+\frac{2ib(\lambda_4^2+\gamma_4^2)^{\frac{1}{2}}}{a[(s+i)^2+4]}P_n'\left(\frac{i\lambda_4}{\gamma_4}\right)\right\} = B_n, \tag{5.6}$$

where $\lambda_4 = \frac{1}{2}(s+i)\,b$ and

$$\frac{2n(n+1)}{2n+1}B_n = \int_{-1}^{+1}(b^2+c^2\mu^2)^{\frac{1}{2}}\,\bar{u}_{4\lambda_1}P_n'(\mu)\,(1-\mu^2)^{\frac{1}{2}}\,d\mu. \tag{5.7}$$

From this point on, the determination of V_4 is in principle straightforward but involves the evaluation of complicated inversion and other integrals. However, since ν is assumed to be small the effect of the tertiary flow may be neglected unless the A_n have poles, *qua* functions of s, in the half plane $\mathscr{R}(s) > 0$, or sufficiently strong singularities on the imaginary axis of s. Our purpose in the rest of this section is to give arguments to exclude such possibilities.

First we show that A_n has no singularities in the half plane $\mathscr{R}(s) > 0$. A singularity can arise if either the coefficient of A_n in (5.6) vanishes or if B_n is singular, or if (2.18) breaks down. Taking these possibilities in order, the coefficient of A_n vanishes if

$$(s-i)\,bP_n'(\psi)+2i\gamma_4 n(n+1)\,P_n(\psi) = 0, \tag{5.8}$$

using the properties of Legendre polynomials and writing

$$\psi = i(s+i)\,b/[4a^2+c^2(s+i)^2]^{\frac{1}{2}}. \tag{5.9}$$

Writing $s = i\sigma$, (5.8) reduces to

$$F(\sigma) \equiv (1-\sigma)\,\psi P_n'(\psi)+(1+\sigma)\,n(n+1)\,P_n(\psi) = 0, \tag{5.10}$$

where

$$\psi = -(1+\sigma)\,b/[4a^2-c^2(1+\sigma)^2]^{\frac{1}{2}}; \tag{5.11}$$

we shall now show that the roots of (5.10) occur at real values of σ. We observe that physically this result is to be expected because it means that the periods of free oscillation of the fluid in the spheroidal envelope are real. In order to prove this result, we note that (5.10) is effectively a polynomial of degree $n+1$ in σ and hence it is enough to prove that $F(\sigma)$ has $n+1$ zeros on the real axis of σ. One such zero is clearly at $\sigma = -1$, where $\psi = 0$. Further, between $\sigma = -1$ and $\sigma = +1$, where $\psi = 1$, P_n has $\frac{1}{2}n$ zeros and consequently, from the interlacing property of the zeros of P_n and P_n', $F(\sigma)$ has $\frac{1}{2}n-1$ zeros between the first positive zero of P_n, *qua* function of ψ, and $\sigma = 1$. A similar remark applies to $-3 < \sigma < -1$. Again if $\frac{1}{2}n$ is even, $F(\sigma) > 0$ if $\sigma+1$ is small and positive, and is negative at the first zero of P_n for which $\sigma > -1$. Hence $F(\sigma)$ vanishes in this range too. A similar argument applies if $\frac{1}{2}n$ is odd. Counting up we see that there are exactly $n+1$ zeros of $F(\sigma)$ in $-3 < \sigma < 1$, all simple, and leading to finite oscillations of the fluid except when they coincide with poles of B_n.

Now consider the singularities of B_n. There are two simple poles at $s = 0$ and at $s = ik$, from (3.8) and (4.10). In addition δ_1 and δ_2 vanish within the range $-3i < s < i$ for any acceptable value of μ. In the evaluation of B_n we need the weighted integral of $\bar{u}_{4\lambda_1}$; consequently the associated singularities of B_n only occur at $s = -3i$ and $s = i$, and will be considered at the same time as the singularities of the governing equation (2.18). The poles of B_n at $s = 0$ and $s = ik$ can lead to motions which are apparently large as $t \to \infty$ if one of the zeros of (5.8) occurs at the same values of s. The second possibility ($s = ik$) is unlikely if $n \neq 2$ since it implies that two free periods of oscillation of the fluid are equal, but we have not been able to rule this out. The first possibility ($s = 0$) is real, and there seems little doubt for any n we can choose a value of a/b to satisfy (5.8) by $s = 0$; for example, if $n = 4$ and

$$6a = (\sqrt{39}+\sqrt{15})\,b,$$

the left-hand side of (5.8) vanishes at $s = 0$. Using the argument above for $s = ik$, $k = (a^2-b^2)/(a^2+b^2)$, we can say that if $a = b$ it is most unlikely that (5.8) is satisfied by $s = 0$ for $n \neq 2$. In the case of particular interest therefore, when $a \approx b$, we conclude that the possibility of satisfying (5.8) at $s = 0$ is not likely to be serious except if $n = 2$.

Let us consider the case $n = 2$ in some detail. Here A_2 has a double pole at $s = ik$ because the coefficient of A_2 vanishes and because B_2 has a simple pole from (4.10). Using the formula for A in (3.5)

$$B_2 = \frac{5}{4}\int_{-1}^{1} \mu(1-\mu^2)^{\frac{1}{2}}\,(b^2+c^2\mu^2)^{\frac{1}{2}}\,\bar{u}_{4\lambda_1}\,d\mu \tag{5.12}$$

$$= -\frac{5A(s-i)}{4b}\int_{-1}^{1} d\mu\,\frac{(b^2+c^2\mu^2)^{\frac{1}{2}}}{a\delta_2}\,[a\mu+(b^2+c^2\mu^2)^{\frac{1}{2}}]\,[\mu(b^2+c^2\mu^2)^{\frac{1}{2}}-a(1-2\mu^2)]. \tag{5.13}$$

The corresponding value of A_2 is now easily worked out from (5.6) and, reverting to the co-ordinates (r,θ,z), the corresponding contribution to $\overline{V}_4$ is

$$\frac{3rz(s^2+1)\,B_2}{a^2(s-i)+b^2(s+i)}. \tag{5.14}$$

Let the coefficient of A in (5.13) be χ. The implication of (5.14) is that the secondary motion described by (3.3), and (3.5) gives rise to a complicated tertiary motion of which the second harmonic is of the same form as (3.3) but with a double pole at $s = ik$. Although formally this means that the tertiary flow increases without limit as $t\to\infty$ it is to be expected that the double pole arises through an error of order χ in the position of the simple pole of A in (3.5). To see this we note that the contribution from (5.14) to the tertiary flow is ultimately dominant and it gives rise via the boundary layer to a similar form to (5.14) in the quaternary flow but now with a triple pole at $s = ik$, and so on. Adding up all these contributions to $\overline{V}$ which are proportional to rz, *qua* functions of r and z, their total is

$$Arz\sum_{m=0}^{\infty}\left[\frac{3(s^2+1)\,\chi}{a^2(s-i)+b^2(s+i)}\right]^m = -\frac{2\Omega a^2 i\sin\alpha(s^2+1)\,rz}{\omega s[a^2(s-i)+b^2(s+i)-3(s^2+1)\,\chi]}. \tag{5.15}$$

Thus the effect of the boundary layer is to shift the pole $s = ik$ to

$$s = ik+\frac{3(1-k^2)\,\chi}{a^2+b^2}+\ldots \qquad \left(k = \frac{a^2-b^2}{a^2+b^2}\right),$$

the neglected terms being $o(\nu^{\frac{1}{2}})$ and most probably $O(\nu^{\frac{3}{4}})$; see Roberts & Stewartson (1963). Of particular interest is the position of the pole when

$$1 \gg \frac{a^2-b^2}{a^2+b^2} \gg \left(\frac{\nu}{\omega a^2}\right)^{\frac{1}{2}}; \tag{5.16}$$

the second condition means that the departures of the surface from the mean sphere, although small, are much larger than the boundary-layer thickness.

So far as χ is concerned it is sufficient to set $a = b$ whence

$$\chi = -\frac{5ai}{4}\int_{-1}^{1} \frac{(1+\mu)^2(1-2\mu)\,d\mu}{\delta_2}, \tag{5.17}$$

where
$$\delta_2 = a^{-1}\left(\frac{\omega a^2}{\nu}\right)^{\frac{1}{2}}(s+i-2i\mu)^{\frac{1}{2}},$$

and since $s \approx 0$

$$\delta_2 = \begin{cases} a^{-1}\left(\dfrac{\omega a^2}{\nu}\right)^{\frac{1}{2}} e^{i\pi/4}(1-2\mu)^{\frac{1}{2}}, & \text{if } \mu < \tfrac{1}{2}, \\ a^{-1}\left(\dfrac{\omega a^2}{\nu}\right)^{\frac{1}{2}} e^{-i\pi/4}(2\mu-1)^{\frac{1}{2}}, & \text{if } \mu > \tfrac{1}{2}. \end{cases}$$

It follows that
$$\chi = \tfrac{5}{4}ia^2\left(\frac{\nu}{2\omega a^2}\right)^{\frac{1}{2}}(0{\cdot}195+1{\cdot}976i), \tag{5.18}$$

whence the pole is moved to

$$s = i\frac{(a^2-b^2)}{(a^2+b^2)} - \left(\frac{\nu}{2\omega a^2}\right)^{\frac{1}{2}}(3{\cdot}720-0{\cdot}365i)+\ldots, \tag{5.19}$$

and, since it is in the half plane $\mathscr{R}(s) < 0$, the associated contribution to the flow dies out as $t \to \infty$. A similar result can be expected for all b/a. We note that the effect of the viscosity on the residue of the pole at $s = 0$ and therefore also on the ultimate motion is small if (5.16) is satisfied, the proportionate changes in the amplitude and orientation of the circulation being $O[\nu/\omega(a-b)^2]^{\frac{1}{2}}$.

It is of speculative interest in connexion with the case $R_1 = 0$ to note that if we set $a = b$ in (5.15) and use the formula for χ in (5.18), then as $t \to \infty$

$$V'' \to 0{\cdot}38\left(\frac{\omega a^2}{\nu}\right)^{\frac{1}{2}}\Omega rz\sin\alpha\cos(\theta-1{\cdot}46)+\Omega r^2\cos\alpha$$

(see (2.13), (2.16): the limit procedures have not been justified). This result suggests that for a sphere the secondary circulation is of the same kind as for a spheroid but the amplitude is $O[\Omega a^2(\omega a^2/\nu)^{\frac{1}{2}}]$ and the orientation is changed.

The last cases to be discussed are the behaviour of the solution in the neighbourhood of $s = \pm i$ and $-3i$, where the differential equation (2.18) takes on a singular form. Although δ_2 vanishes at $\mu a = (b^2+c^2\mu^2)^{\frac{1}{2}}$ when $s = i$ it follows from (4.10) and (5.7) that B_n is bounded as $s \to i$ for all n. Hence, substituting into (5.6) and noting that $\lambda_4 = i\gamma_4$ on S_0 if $s = i$, it follows that

$$A_n = O(s-i)^{\frac{1}{2}}, \tag{5.20}$$

near $s = i$. Further the nth harmonic in (5.4) can be written as

$$\tfrac{1}{4}[(s+i)^2+4]^{\frac{1}{2}}(s+i)\,rzA_n G_{n/2}\{\tfrac{1}{4}[(s+i)^2+4]\,r^2,\ \tfrac{1}{4}(s+i)^2z^2\}, \tag{5.21}$$

where $G_{n/2}(\alpha, \beta)$ is a homogeneous polynomial of degree $\frac{1}{2}n$ in α and β with constant coefficients. Hence the contribution to the tertiary flow is $O(s-i)$ as $s \to i$, and its inverse with respect to s tends to zero as $t \to \infty$. A similar remark applies to the neighbourhood of $s = -3i$. In the neighbourhood of $s = -i$, B_n is $O(s+i)$ from (4.10), and the coefficient of A_n in (5.6) is $O(1)$ since n is even. Consequently (5.21) is $O(s+i)^3$ near $s = -i$ and its contribution to the tertiary flow tends to zero as $t \to \infty$.

Summarizing we can expect that, of all the components of the secondary flow which depend on θ, only the steady component will remain as $t \to \infty$, the boundary layer serving to damp out the oscillatory terms.

6. The tertiary inviscid flow (ii)

In this section we consider the inviscid flow engendered by (4.17) which is independent of θ. We assume that the associated inviscid flow $\mathbf{u}_5$ is also independent of θ whence, on taking the Laplace transform with respect to ωt, the governing equations reduce to

$$\left.\begin{aligned} su^*_{5r} - 2u^*_{5\theta} &= -\frac{\partial V^*_5}{\partial r}, \quad su^*_{5\theta} + 2u^*_{5r} = 0, \\ su^*_{5z} &= -\frac{\partial V^*_5}{\partial z}, \quad \frac{\partial}{\partial r}(ru^*_{5r}) + \frac{\partial}{\partial z}(ru^*_{5z}) = 0; \end{aligned}\right\} \tag{6.1}$$

these equations also follow from (2.12) on assuming that $\mathbf{u}_5$ is independent of θ and neglecting the initial and forcing terms. Further, on S_0,

$$\frac{r}{a^2}u^*_{5r} + \frac{z}{b^2}u^*_{5z} = \left[\frac{r^2}{a^4} + \frac{z^2}{b^4}\right]^{\frac{1}{2}} u^*_{5\lambda_1}(b, \mu), \tag{6.2}$$

from (4.17). In terms of V^*_5,

$$u^*_{5r} = -\frac{s}{s^2+4}\frac{\partial V^*_5}{\partial r}, \quad u^*_{5z} = -\frac{1}{s}\frac{\partial V^*_5}{\partial z}, \tag{6.3}$$

so that

$$\frac{\partial}{\partial r}\left[r\frac{\partial V^*_5}{\partial r}\right] + \frac{(s^2+4)}{s^2}\frac{\partial^2 V^*_5}{\partial z^2} = 0. \tag{6.4}$$

Introduce new co-ordinates μ_5, λ_5 such that

$$r = \frac{2}{s^2+4}(\lambda_5^2 + \gamma_5^2)^{\frac{1}{2}}(1-\mu_5^2)^{\frac{1}{2}}, \quad z = \frac{2}{s}\mu_5\lambda_5, \tag{6.5}$$

where $\gamma_5 = a^2 + \frac{1}{4}s^2c^2$, S_0 is given by $\lambda_5 = \frac{1}{2}b_5$ and, on S_0, $\mu_5 = \mu_4 = \mu_1 = \mu$.

The boundary conditions now reduce to

$$-\frac{\partial V^*_5}{\partial \lambda_5} = a^{-1}(b^2 + c^2\mu^2)^{\frac{1}{2}} u^*_{5\lambda}, \tag{6.6}$$

on S_0. The most general appropriate solution for V^*_5 in S_0 is

$$V^*_5 = \sum_{n=0}^{\infty} C_n P_n\left(\frac{i\lambda_5}{\gamma_5}\right) P_n(\mu_5), \tag{6.7}$$

where C_n are constants and, since $u^*_{5\lambda}$ is an even function of μ, n also must be even. Hence

$$\frac{2}{(2n+1)} C_n P'_n\left(\frac{ib_5}{2\gamma_5}\right) = -\frac{\Omega i \cos\alpha}{2\omega s}\int_{-1}^{+1} P_n(\mu)\frac{\partial}{\partial\mu}\left[(b^2+c^2\mu^2)^{\frac{1}{2}}(1-\mu^2)\left(\frac{1}{\delta_3}-\frac{1}{\delta_4}\right)\right]d\mu. \tag{6.8}$$

The main interest here centres on the behaviour of the solution at large times, contributions to which arise from three sources. To begin with C_n will develop a simple pole whenever P'_n vanishes unless $s = 0$: the corresponding motion, according to (6.7), is a feeble oscillation and one can expect to show, using a similar argument to that which led to (5.19), that viscosity damps it out ultimately. Further the differential equation is singular at $s = \pm 2i$ where $ibs/2\gamma_5 = \pm 1$. Since $P'_n(\pm 1) \neq 0$, the leading terms from this cause must become zero, as $\rightarrow \infty$. Finally, one must consider the case $s = 0$. When s is very small

$$C_n = -\frac{(2n+1)\,\Omega\cos\alpha a^3}{n(n+1)\,b\omega s^2 P_n(0)}\left(\frac{\nu}{\omega a^2}\right)^{\frac{1}{2}}\int_{-1}^{+1} d\mu(1-\mu^2)\,P'_n(\mu)\,\frac{(b^2+c^2\mu^2)^{\frac{1}{4}}}{(-2i\mu a)^{\frac{1}{2}}}, \tag{6.9}$$

so that, on inverting (6.7), V^*_5 contains a term which is proportional to t when t is large. The implication is that the boundary layer exerts a decisive influence on the inviscid flow outside in this instance.

The reason for the difference between the solution in § 5 near $s = -i$ and the solution in § 6 near $s = 0$, although both governing equations are similar and singular at these points is two-fold. First (4.10) contains a factor $(s+i)$ while (4.17) contains a factor s^{-1} so that the behaviour of V^*_5 near $s = 0$ is bound to be more singular than the behaviour of V^*_4 near $s = -i$. There is, however, a second deeper reason. Near $s = 0$ the governing equation (6.1) reduces to

$$u^*_{5r} = 0, \quad \frac{\partial u^*_{5z}}{\partial z} = 0, \tag{6.10}$$

which means that u^*_{5z} is independent of z and therefore must be an even function of μ. Consequently V^*_5 is an odd function of μ, but the boundary condition (6.2) requires it to be an even function. This patent contradiction means that a serious breakdown in the solution must occur as $t \rightarrow \infty$. From studies of the steady state (e.g. Proudman 1956) we know that the boundary layer in such a case exerts a decisive influence on the flow outside, adjusting it until the condition that $u_{5\lambda}$ is an odd function of μ is satisfied, which is only possible if $u_{5\lambda} \rightarrow 0$ at $t \rightarrow \infty$. Hence the initial angular velocity $u_\theta = \Omega r \cos\alpha$ is also damped out by the boundary layer.

On the other hand, near $s = -i$, while the governing equations (2.17) and (2.18) also reduce in part to

$$\frac{\partial u^*_{4z}}{\partial z} = 0, \tag{6.11}$$

so that u^*_{4z} is an odd function of μ, this condition is actually satisfied by (4.10) so that no corresponding difficulties occur.

7. Brief summary: geophysical application

Our analysis has established that, no matter what the initial motion of the fluid in the cavity may be, if

$$R_1 = \frac{(a^2-b^2)\,\omega}{\Omega a^2}, \quad R_1 R_2 = \frac{\omega(a^2-b^2)}{\nu}, \quad R_3 = \frac{\omega a^2}{\nu}$$

are all large, the flow will ultimately (i.e. in a time of order a^2/ν) be the same. It may be described most easily in a frame of reference rotating about the axis of precession with the angular velocity Ω of precession. The primary flow is then merely a solid-body rotation of the fluid and envelope together. Superimposed upon this is a steady flow which, apart from a boundary layer whose thickness is nearly everywhere of order $(\nu/\omega)^{\frac{1}{2}}$ within which the fluid adjusts to the no-slip conditions at the interface, is characterized by closed elliptical streamlines and by uniform vorticity

$$\frac{2(a^2+b^2)}{\omega^2(a^2-b^2)}\,\boldsymbol{\omega}\times(\boldsymbol{\Omega}\times\boldsymbol{\omega}). \tag{7.1}$$

The streamlines lie in planes perpendicular to this, and are similar to the elliptical section of the boundary by a plane through the axis of symmetry. In fact, if y be a Cartesian co-ordinate drawn along $\boldsymbol{\omega}\times\boldsymbol{\Omega}$, z along $\boldsymbol{\omega}$ and x drawn to complete the triad, this flow is given by

$$\frac{2\Omega a^2 b^2 \sin\alpha}{(a^2-b^2)}\left(0, \frac{z}{b^2}, -\frac{y}{a^2}\right). \tag{7.2}$$

Its greatest magnitude (taken at the poles) is

$$U = 2\Omega a^2 b \sin\alpha/(a^2-b^2). \tag{7.3}$$

We will now amplify the brief remarks given in the Introduction on the geophysical relevance of the present theory to the hydrodynamics of the Earth's core. Let us first briefly estimate the parameters R_i. Many calculations have been made of the ellipticity ϵ of the equidensity surfaces within the Earth. They have been generally based on Clairaut's equation (cf. e.g. Chandrasekhar & Roberts 1963), and have led to the conclusion that ϵ decreases monotonically from the value of 1/297 for the Earth's surface to a value of approximately 1/470 at the centre of the Earth (e.g. Lambert & Darling 1951). In particular, a computation of the ellipticity of the core has given $\epsilon = 1/390$, i.e.

$$\frac{(a^2-b^2)}{a^2} \doteqdot 2\epsilon \doteqdot 0{\cdot}00513. \tag{7.4}$$

Now for the luni-solar precession

$$\alpha = 23{\cdot}452^\circ \quad \text{and} \quad 2\pi/\Omega = 25{,}725\,\text{years}, \tag{7.5}$$

thus

$$\omega/\Omega = 9{\cdot}40\times 10^6. \tag{7.6}$$

It follows that

$$R_1 = 4{\cdot}82\times 10^4. \tag{7.7}$$

The kinematic viscosity of the Earth's core is notoriously difficult to estimate (e.g. Hide & Roberts 1961), and values between 10^{-3} cm^2/sec and 10^8 cm^2/sec have been offered, of which the latter is certainly an extreme upper limit. Even this value, however, implies that both $R_1 R_2$ and R_3 exceed unity; in all probability, therefore, they do so by large factors. Thus, the basic tenets of the analysis presented in the earlier sections are satisfied, and its particular relevance to theories of the geomagnetic field can be legitimately examined.

It is today generally accepted that the source of the geomagnetic field is to be found in a self-excited dynamo located deep within the Earth, the 'moving parts' being the hydrodynamic motions of the core. Naturally, if a driving mechanism were not available, Ohmic dissipation would cause the dynamo to run down in an electromagnetic decay time ($\sim$ 10,000 years). On the other hand, palaeomagnetic studies have established that the magnitude of the geomagnetic field has not varied greatly over geological time. Evidently, then, a capital question is to determine the nature of the driving mechanism which must be present. In an early study, Bullard (1949) singled out thermal convection as the most likely cause. He was, however, unable to exclude the possibility that the dynamo is precessionally driven. Indeed, the subsequent analysis by Bondi & Lyttleton (1953), though inconclusive, drew attention to the singularities of the boundary layer at latitudes 30° N. and 30° S. (cf. § 4 above) as a possible source of instabilities which would intermittently shed turbulent eddies. These, the authors felt, would through their inductive effects contribute to the secular variation of the geomagnetic field in these latitudes. They might also explain the observed small irregular fluctuations in the Earth's state of rotation. It is fair, therefore, to state that their ideas enhanced, if anything, the interest in the precessional effects. However, unsteady motion was invoked by Bondi & Lyttleton to meet a situation for which, they maintained, a steady solution did not exist. In our model, on the other hand, a steady flow does establish itself, and there seems to be no obvious reason why it should be unstable or be associated with the observed irregular fluctuations, either in the geomagnetic field or in rotation.

Bullard called attention to early studies of precessionally driven flows within a spheroidal cavity, and in particular ascribed to Poincaré (1910) a theorem that 'the ellipticity of the earth is sufficient to ensure that the material of the core moves with the rest of the earth like a rigid body in a small motion of precession, even in the absence of viscosity'. Again, he comments 'It has not been proved that the motion...is possible for a precession of finite amplitude (the angle at the vertex of the core swept out by the earth's axis is 47°)'. It seems to us, however, that Poincaré's solution, like ours, depends only on the assumption that $\mathbf{u}$ and $\mathbf{\Omega}$ are small; it does not require that α, too, is small. His solution, like ours, consists of a primary solid-body rotation with the mantle upon which a steady secondary flow of uniform vorticity is superimposed. (He did not, however, demonstrate that this solution is the one realized in practice in the limit $\nu t \to \infty$.) Bullard did not discuss the secondary flow. It should be observed, however, that its magnitude (cf. equation (7.3)) is of the order of 1 cm/sec. The horizontal velocities near the surface of the core, as inferred by studies of the westward drift, are also of this order of magnitude (or even rather smaller). Thus the precessional flow should be quite large enough to make its inductive effects felt; assuming an electrical conductivity σ of 3×10^5 mho/m, the corresponding magnetic Reynolds number R_m is

$$R_m = \mu_0 \sigma a U \doteq 1{\cdot}3 \times 10^4 \gg 1 \qquad (7.8)$$

(μ_0 is the permeability of free space, $4\pi \times 10^{-7}$ Henry/m). For dynamo action it is necessary that R_m should, indeed, be large. However, it is also necessary that the motions have a sufficiently low degree of symmetry, and it seems to us that, in this respect, the motion (7.2) would be incapable of amplifying and maintaining a stray magnetic field.

Finally one further point should be noted. The theory we have described has ignored Lorentz forces. This is appropriate when the solution is used to determine whether precessional flows can, by themelves, amplify a stray weak magnetic field. There is some difficulty, however, in applying it without modification to the actual geophysical situation in which strong magnetic fields are present. The precessional flows we have described may be thought of as being driven by a force per unit mass of approximately $\omega \Omega a$, i.e. about 10^{-7} cm^2/sec. The Lorentz forces per unit mass $B^2/\mu_0 \rho a$ may well be larger; indeed, if we assume in the core a geomagnetic field B of only 50 G, their magnitude is 10^{-7} cm^2/sec. The toroidal field is, however, probably considerably in excess of 50 G. Further, since the lower mantle is appreciably conducting, there will be a magnetic coupling of core and mantle; similar order of magnitude estimates may be made, and indicate that this effect, is likely to dominate the viscous coupling.

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